A New Fast and Accurate Algorithm for the Computation of Microstrip Capacitances

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Abstract—**A new algorithm is presented for the calculation of TEM parameters of microstrips, based on a lumped-circuit model of the microstrip transverse static field. The use of such an algorithm allows one to obtain either very high accuracies or very low computer times when ordinary accuracies are tolerated. Compared with the similar technique by Lennartsson [19], the analysis method presented here allows substantial reduction in computer time, storage, and errors.**

I. INTRODUCTION

MICROSTRIP circuit design requires efficient algorithms for the computation of the line-per-unit-length static parameters, knowledge necessary both in the TEM approximation and for taking into account the dispersion effects. While in current literature conformal mapping [1], [2], variational techniques [3]–[8], the relaxation algorithm [9]–[11], the method of subareas [12], the method of Green's function [13]–[16], or its varieties like the multiple image technique [17] and moment method [18] are generally used, this paper presents an algorithm based only on well-known concepts of lumped-circuit theory.

The main features of this algorithm are as follows.

1) The lumped model of the distributed capacitance from which all other static parameters can be derived, drastically reduces the number of loops and nodes of the circuit with respect to the classical discrete model used in [19]. Moreover, the discretization is used in only one dimension, parallel to the substrate.

2) Matrix inversion, used in most of the known methods, is avoided in favor of an algorithm due to Levinson [20] which allows remarkable reductions of storage, computer time, and roundoff errors.

3) A formula indicates the influence of the discretization errors on the capacitance value: this allows a further time reduction.

4) In the synthesis problem, the algorithm is very convenient since the capacitance values, for different values of the strip width, are obtained as partial results without a further computation.

5) The program works on a small computer IBM 1130 with 8K storage. On a computer IBM 360/67 with double precision, an excellent accuracy (10^{-8}) or, alternatively, a very low computation time (0.02 s for a capacitance calculation) can be attained.

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II. LUMPED MODEL

The aim of this paper is the computation of the per-unit-length capacitance of an open microstrip, with strip width w and substrate thickness h (Fig. 1), by using a lumped equivalent circuit (Fig. 2), which is obtained by discretizing Laplace's partial differential equation. The discretization is realized with N steps of length u along the ξ axis (parallel to the ground) and with an infinite number of steps of length αu along the orthogonal η axis. Between the geometrical dimensions of the microstrip w and h and the parameters M and H of the model, the following relations hold:

$$M = w/u \quad H = h/(\alpha u) - 1. \tag{1}$$

Furthermore, in the model, the width of the substrate is not infinite, as theoretically supposed; however, we assume that the ratio $K = uN/h$ is large enough, hence also $uN/w \gg 1$, in such a way that the error is small; later on this error will be drastically reduced.

The values, normalized with respect to ϵ_0 (free-space permittivity), of the elementary capacitances are linear

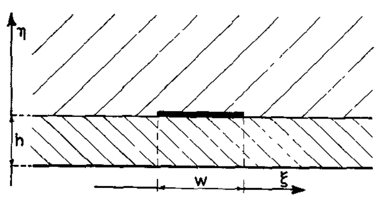

Fig. 1. Open microstrip.

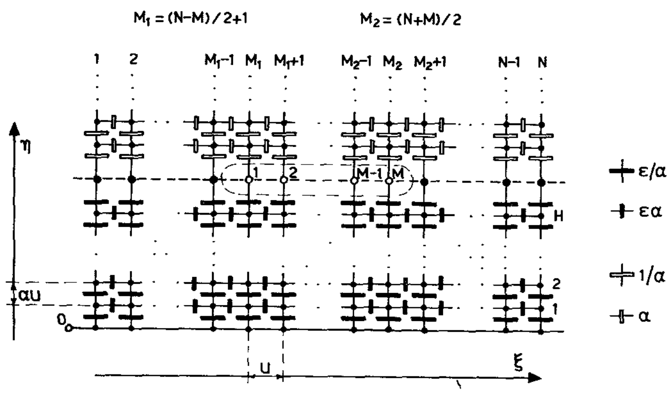

Fig. 2. Lumped model of the distributed capacitance.

or inverse linear functions of α; ϵ is the relative permittivity of the substrate (Fig. 2). Later on, α will tend to zero; that is, the discretization along the η axis will be avoided in order to reduce the errors and to simplify the formulas. In Fig. 2 the upper strip is indicated by a dotted line which encircles the M terminals corresponding to it.

Thus it is necessary to compute the $M \times M$ impedance matrix Z_M of the terminals $1,2,\cdots,M$ with respect to node 0, which represents the microstrip ground in the model. Then the microstrip capacitance can be found by short-circuiting the terminals $1,2,\cdots,M$, and by computing the admittance between the resulting node and node 0.

If the model of Fig. 2 were directly used as in [19], the computation of Z_M would be rather formidable, while the circuit transformations, shown in the next section, will allow closed formulas for the elements of Z_M to be obtained.

III. CIRCUIT TRANSFORMATIONS

The N-port of Fig. 2, with the nodes $0,1,2,\cdots,M$ as output terminals, becomes much more simplified with suitable equivalent transformations. The following matrix, representing the short-circuit admittance of a degenerate uniform ladder structure, whose shunt branches are open circuits, is first diagonalized [21] by a similarity transformation:

$$G_N = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} = T_N G_{N0} T_N^{-1} \quad (2)$$

where $T_N (T_N^t = T_N^{-1})$ and G_{N0} are the orthogonal modal matrix and the spectral representation of G_N, respectively. Their elements are, respectively,

$$\left.\begin{aligned} t_{ij} &= (2/N)^{1/2} \cos\left[\frac{\pi(j-1)(i-0.5)}{N}\right], & j &= 2,3,\cdots,N \\ t_{i1} &= 1/(N)^{1/2} & i &= 1,2,\cdots,N \end{aligned}\right\} \quad (3)$$

$$g_n = 2\left[1 - \cos\left(\frac{\pi(n-1)}{N}\right)\right], \qquad n = 1,2,\cdots,N. \quad (4)$$

The definition of the square root b_n of g_n will be useful later on

$$b_n = g_n^{1/2} = 2\sin\left[\frac{\pi(n-1)}{2N}\right], \qquad n = 1,2,\cdots,N. \quad (5)$$

By the preceding diagonalization, the circuit of Fig. 2 can be transformed into that of Fig. 3(a): actually the insertion of four $2N$-port transformers Γ_1, Γ_2, Γ_3, and Γ_4 allows the decomposition of the capacitance reticulum of Fig. 2 in $2N$ ladder 2-ports $\Lambda_n^{(h)}$, $\Lambda_n^{(\infty)}$ $(n = 1,2,\cdots,N)$, whose values are determined by the g_n elements defined in (4). The N two-ports $\Lambda_n^{(h)}$ are composed of $H+1$ series branches and of H shunt branches [Fig. 3(b)], while the N 2-ports $\Lambda_n^{(\infty)}$ are similar, but with an infinite number of branches and with a higher impedance level [Fig. 3(c)]. However, $\Lambda_1^{(h)}$ and $\Lambda_1^{(\infty)}$ are degenerate ladder networks with shunt branches open, since $g_1 = 0$. That is, $\Lambda_1^{(h)}$ is a single admittance $y_1 = \epsilon/h$ [Fig. 3(d)], while $\Lambda_1^{(\infty)}$ is an open-circuited 2-port.

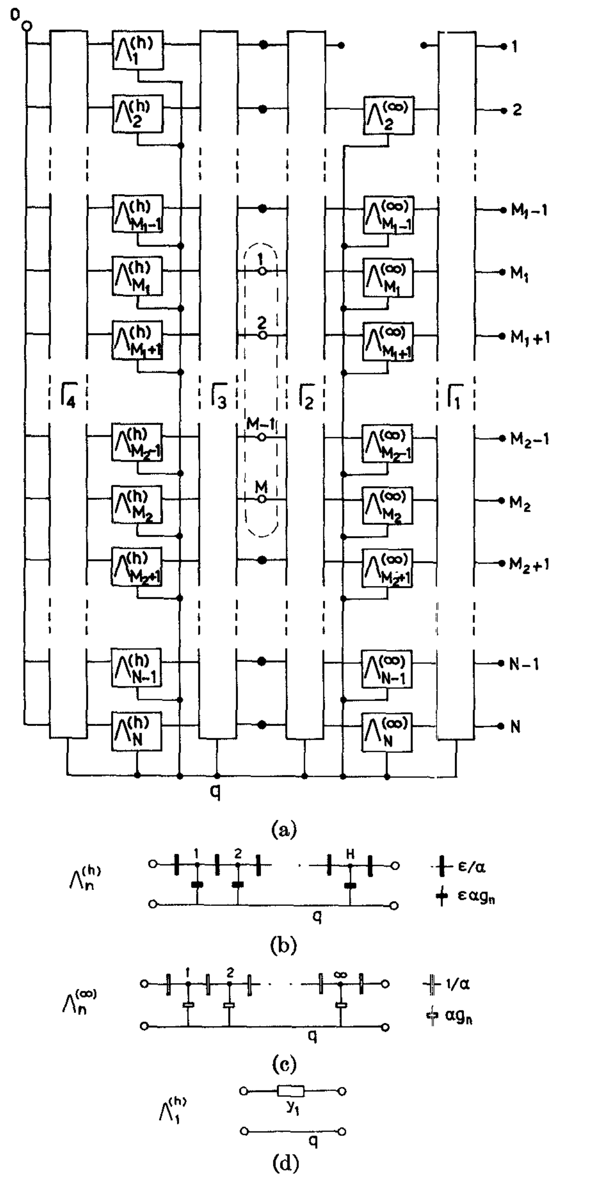

Fig. 3. (a) First equivalent circuit of the lumped model of Fig. 2. (b) 2-ports $\Lambda_n^{(h)}$ $(n = 2,\cdots,N)$. (c) 2-ports $\Lambda_n^{(\infty)}$ $(n = 2,3,\cdots,N)$. (d) 2-port $\Lambda_1^{(h)}$.

As mentioned in Section II, the discretization along the η axis can be avoided by letting α go to zero in the elementary capacitances of $\Lambda_n^{(h)}$ and $\Lambda_n^{(\infty)}$ [Fig. 3(b) and (c)]. Their characteristic admittance and characteristic propagation factors become

$$y_{0n}^{(h)} = \epsilon b_n \quad \vartheta_{0n}^{(h)} = h b_n,$$
$$y_{0n}^{(\infty)} = b_n \quad \vartheta_{0n}^{(\infty)} = \infty, \qquad n = 2,3,\cdots,N \tag{6}$$

whereas the chain matrices of Γ_2, Γ_4 and Γ_1, Γ_3 are, respectively, [Fig. 3(a)]

$$\begin{bmatrix} T_N & 0_N \\ 0_N & T_N \end{bmatrix} \qquad \begin{bmatrix} T_N^{-1} & 0_N \\ 0_N & T_N^{-1} \end{bmatrix}. \tag{7}$$

Thus Γ_2 and Γ_4 are mirror images equal to Γ_1 and Γ_3.

Furthermore, in Fig. 3(a) the reference node q, common to the 2-ports and to the transformers, does not at all coincide with the microstrip ground (node 0).

Now the circuit of Fig. 3(a) can be further simplified. Actually Γ_2 and Γ_3 can be substituted by a single transformer [Γ_3 in Fig. 4(a)] because they can be considered in parallel since they are mirror image equal. Then Γ_1 can be avoided because the $N-1$-ports $\Lambda_n^{(\infty)}$ connected with it have infinite attenuation [see (6)]. The last Γ_4 can be substituted by a 2-port transformer Γ_5 [Fig. 4(a)] with voltage ratio $N^{1/2}$ since any of its input terminals are connected with the ground (node 0).

Furthermore the $N-1$ single driving point admittances $y_n (n = 2,3,\cdots,N)$ can substitute the $2N-2$ 2-ports $\Lambda_n^{(h)}$ and $\Lambda_n^{(\infty)}$ [Fig. 4(a)]; any admittance y_n is composed of a connection of two 2-ports $\Lambda_n^{(h)}$ and $\Lambda_n^{(\infty)}$ [Fig. 4(b); see (6)]

$$y_n = b_n[1 + \epsilon/th\ (hb_n)], \qquad n = 2,3,\cdots,N. \tag{8}$$

Finally the transformer Γ_5 can be eliminated, obtaining the circuit of Fig. 5, where the admittance y_1' is

$$y_1' = Ny_1 = \epsilon N/h. \tag{9}$$

IV. IMPEDANCE MATRIX

A physical consideration simplifies the computation of matrix $Z_M(z_{m_1,m_2})$: since the width of the whole layer Nu is much larger than the width w of the upper strip (Fig. 1), Z_M becomes a symmetrical Toeplitz matrix, where any element depends only on the difference of the two indices

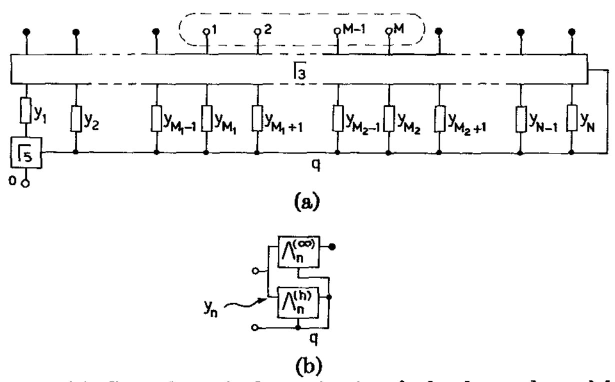

Fig. 4. (a) Second equivalent circuit of the lumped model of Fig. 2. (b) Driving point admittances y_n ($n = 2,3,\cdots,N$).

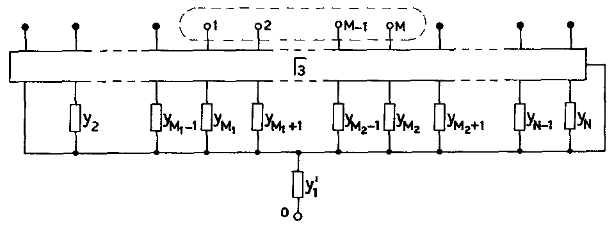

Fig. 5. Third (definitive) equivalent circuit of the lumped model of Fig. 2.

$$z_{m_1,m_2} = R_m^{(N)} \quad \text{with} \quad m = 1 + |m_1 - m_2|,$$
$$m = 1,2,\cdots,M. \tag{10}$$

The index N denotes the considered width of the substrate.

This property of Z_M allows a remarkable reduction of computer time and storage because only M elements of Z_M must be computed, and also because it can be inverted with a more convenient algorithm than the current ones, as will be seen later on.

Hence the nodes m_1 and m_2 must be chosen in the following manner:

$$m_1 = (N - m + 2)/2$$
$$m_2 = (N + m)/2, \qquad N + m \text{ even}$$
$$m_1 = (N - m + 1)/2$$
$$m_2 = (N + m - 1)/2, \qquad N + m \text{ odd}. \tag{11}$$

For example, if $m = 1$, m_1 is equal to m_2, that is, $R_1^{(N)}$ is the driving-point impedance between the node $N/2$ (N even) or $(N+1)/2$ (N odd) and the node 0 (Fig. 5).

Now we investigate the analytical expression of $R_m^{(N)}$ ($m = 1,2,\cdots,M$) by inspection of the equivalent circuit of Fig. 5 and by the definition (10)

$$R_m^{(N)} = h/(\epsilon N) + \sum_{n=2}^{N} [(t_{m_1,n} t_{m_2,n})/y_n],$$
$$m = 1,2,\cdots,M \tag{12}$$

where the product $t_{m_1,n}t_{m_2,n}$ can be developed from (3)

$$t_{m_1,n}t_{m_2,n} = (-1)^{n-1}/N + [(-1)^n - (-1)^{N+m+n}]b_n^2/(4N) + \cos[\pi(n-1)(m-1)/N]/N. \tag{13}$$

Thus we can split (12) by using (13)

$$R_m^{(N)} = R_m^* + R_a, \qquad m = 1,2,\cdots,M \tag{14}$$

where

$$R_a = 0 \quad \text{for} \quad N + m \text{ even}$$
$$R_a = r_a = \sum_{n=2}^{N} [(-1)^n b_n^2/(2Ny_n)] \quad \text{for} \quad N + m \text{ odd} \tag{15}$$

$$R_m^* = \sum_{n=2}^{N} \{\cos[\pi(n-1)(m-1)/N]/(Ny_n)\} + h/(\epsilon N) - \sum_{n=2}^{N} [(-1)^n/(Ny_n)]. \tag{16}$$

Now let us define

$$r_m = R_m^* - R_{m-1}^*, \qquad m = 1,2,\cdots,M \tag{17}$$

where $R_0^* = 0$ for sake of convenience.

Then from (16) and (17) we obtain

$$r_m = -\sum_{n=2}^{N} \{\sin[\pi(n-1)(m-1.5)/N]b_n/(Ny_n)\}, \qquad m = 2,3,\cdots,M \tag{18}$$

for $m = 1$

$$r_1 = R_1^* = h/(\epsilon N) + \sum_{n=3,5,7\cdots}^{N} [2/(Ny_n)]. \tag{19}$$

Now we could compute the impedance $R_m^{(N)}$, but the sums present in (15), (18), and (19) are extended to very many terms because N is generally very high. A suitable investigation of this problem allows a computation reduction; the factor $1/(Ny_n)$ [see (8)] can be split into two addends

$$1/(Ny_n) = d/[b_n(e^{2hb_n} + k)] + f/b_n, \quad n = 2,3,\cdots,N \tag{20}$$

where

$$d = -2\epsilon/[N(1+\epsilon)^2] \quad k = (\epsilon-1)/(\epsilon+1)$$
$$f = 1/[N(1+\epsilon)]. \tag{21}$$

$$r_1' = h/(\epsilon N) + f\sum_{n=3,5,7}^{N_s} 2/b_n + \ln \frac{\sin[\pi(N+N_s^*-2)/(4N)] + \sin[\pi(N-N_s^*)/(4N)]}{\sin[\pi(N+N_s^*-2)/(4N)] - \sin[\pi(N-N_s^*)/(4N)]} \Big/ [\pi(1+\epsilon)]. \tag{34}$$

Now we can substitute (20) into (15), (18), and (19)

$$r_a = r_a' + r_a'', \tag{22}$$

$$r_m = r_m' + r_m'', \qquad m = 1,2,\cdots,M \tag{23}$$

where the following equalities hold:

$$r_a' = (f/2)\sum_{n=2}^{N} [(-1)^n b_n] \tag{24}$$

$$r_a'' = (d/2)\sum_{n=2}^{N_s} [(-1)^n b_n/(e^{2hb_n} + k)] \tag{25}$$

$$r_m' = -f\sum_{n=2}^{N} \sin[\pi(n-1)(m-1.5)/N], \qquad m = 2,3,\cdots,M \tag{26}$$

$$r_m'' = -d\sum_{n=2}^{N_s} \{\sin[\pi(n-1)(m-1.5)/N]/(e^{2hb_n} + k)\}, \qquad m = 2,3,\cdots,M \tag{27}$$

$$r_1' = h/(\epsilon N) + f\sum_{n=3,5,7\cdots}^{N} (2/b_n) \tag{28}$$

$$r_1'' = d\sum_{n=3,5,7\cdots}^{N_s} \{2/[b_n(e^{2hb_n} + k)]\}. \tag{29}$$

The upper bound N_s of the sums in (25), (27), and (29) is theoretically equal to N, but in practice it can be chosen much lower than N, because the factor $1/(e^{2hb_n} + k)$ converges to zero for increasing n. The value N_s can be evaluated from

$$e^{4h}\sin[\pi(N_s-1)/(2N)] \simeq 10^{\delta} \tag{30}$$

i.e.,

$$N_s \simeq N\delta \log_e 10/(2h\pi) \tag{31}$$

where $10^{-\delta}$ is the required precision.

The summations in (24), (26), and (28) can be substituted by suitable trigonometric formulas, i.e.,

$$r_a' = f\sin\left[\frac{\pi(N-1)(2N+1)}{4N}\right]\sin[\pi(N/2) - 3\pi/4] \Big/ \sin[\pi/2 + \pi/(4N)] \tag{32}$$

$$r_m' = -f\sin\left[(N-1)\frac{\pi}{2N}(m-1.5)\right] \cdot \sin\left[\frac{\pi}{2}(m-1.5)\right] \Big/ \sin\left[\frac{\pi}{2N}(m-1.5)\right] \tag{33}$$

where $N_s^* = N_s$ or $N_s^* = N_s + 1$ for N_s even or odd, respectively. At this point one has the complete formulas necessary for the computation of the vector $R_m^{(N)}$, i.e.,

$$R_1^{(N)} = r_1' + r_1'' + R_a$$
$$R_{m+1}^{(N)} = R_m^{(N)} + R_a + r_{m+1}' + r_{m+1}'' \tag{35}$$

$R_a = 0$ for $N + m$ even, $R_a = r_a' + r_a''$ for $N + m$ odd.

Now it is necessary to compute the values of $R_m^{(N)}$ $(m = 1,2,\cdots,M)$ for $N = \infty$. On the basis of several numerical examples, the following recursive formula can be shown to hold:

$$R_m^{(N)} \simeq R_m^{(N/2)} - \Delta_m/N^2, \qquad m = 1,2,\cdots,M \tag{36}$$

where Δ_m in a good approximation does not depend on N. If

$$\mathbf{R}_m^{(N)} = [4R_m^{(N)} - R_m^{(N/2)}]/3 \tag{37}$$

then it is easy to prove that $\mathbf{R}_m^{(N)}$ approximates the value $R_m^{(\infty)}$ much more closely than $R_m^{(N)}$. Thus it is possible to have excellent approximation with N not too large.

V. MATRIX INVERSION

Now the terminals $1,2,\cdots,M$ must be connected in only one node in order to find the capacitance C with respect to node 0; actually it is

$$C = [1,1,\cdots,1]Z_M^{-1}[1,1,\cdots,1]^t \tag{38}$$

TABLE I

K \ M	4	8	16	32	64	128	256
4	12.832833014	12.778402485	12.778806580	12.778303290	12.778392201	12.778396426	12.778397004
8	12.502125223 $4\ 10^{-3}$	12.484408741	12.483317026	12.483230474	12.483219221	12.483219017	12.483219006
16	12.470526891	12.456365342 $1.6\ 10^{-4}$	12.455133715	12.454969028	12.454995091	12.454989310	12.454989226
32	12.469576405	12.455505540	12.454220597 $1.2\ 10^{-5}$	12.454125746	12.454117502	12.454116658	12.454116570
64	12.469530228	12.455464747	12.454178043	12.454083476 $0.9\ 10^{-6}$	12.454074458	12.454074146	12.454073973
128	12.469527464	12.455462023	12.454175199	12.454080726	12.454072476 $1.0\ 10^{-7}$	12.454071503	12.454071490
256				12.454080582	12.454072237	12.454071432 $1.0\ 10^{-8}$	12.454071334
512				12.454080571	12.454072234	12.454071412	12.454071323 $1.0\ 10^{-9}$

Note: Values of $\mathbf{C}\ (w/h,\epsilon,M,K)$ (per-unit-length capacitances of microstrip, normalized with respect to ϵ_0) as functions of K and M, while w/h and ϵ are equal to 1 and 6, respectively. For $K = 2M$ the relative errors with respect to $\mathbf{C}$ (1,6,256,512) are reported under the preceding values. The relative error of $\mathbf{C}$ (1,6,256,512) is estimated by extrapolation.

that is, C is the sum of the M^2 elements of the matrix Z_M^{-1}. Since Z_M is a Toeplitz matrix [see (10)] the iterative Levinson algorithm [20] allows C to be computed, obtaining as partial results the sums C_m of the m^2 elements of the matrices Z_m^{-1} $(m = 1,2,\cdots,M-1)$, where Z_m is the submatrix of Z_{m+1}, obtained by deleting the last row and the last column, i.e.

$$C_m = [1,1,\cdots,1]Z_m^{-1}[1,1,\cdots,1]^t. \tag{39}$$

Hence it will be $C = C_M$.

This algorithm has the following advantages compared to classical matrix inversion methods: reduced storage, low roundoff errors, and very short computer time.

A small computer like the IBM 1130 with 8K storage, allows the inversion of a 300×300 Toeplitz matrix; thus very dense discretization and/or very short computer time are obtained.

But this algorithm has another characteristic; the partial results C_m [see (39)] are the values of the capacitances of the microstrip with width of the upper strip (see Figs. 1 and 2) equal to wm/M and with the same thickness h. Then by only one step one can find the whole curve of C_m as function of wm/M for $m = 1,2,\cdots,M$. This is very important in the synthesis problem because one can immediately obtain the value of w corresponding to the required value of capacitance by a direct interpolation.

VI. ERRORS AND COMPUTER TIMES

The numerical results of the above algorithm are affected by three types of errors:

- e_1 error due to noninfinity of M, i.e., due to discretization;
- e_2 error due to noninfinity of $K = Nu/h$;
- e_3 error due to finite word length, i.e., due to roundoff.

If C is the effective value and C_0 is the computed value, then

$$C(w/h,\epsilon) = C_0(w/h,\epsilon,M,K) + e_1(w/h,\epsilon,M) + e_2(w/h,\epsilon,K) + e_3. \tag{40}$$

Error e_3 is stochastic and can be neglected for the moment since the considered values of M and K are such that the deterministic error e_1 is prevailing; e_2 is also negligible if the formula (37) is used with $K = 8 + 32$. Only e_1 is a significant error, but it can be estimated on the basis of a great number of numerical examples; it turns out that it can be represented by the approximate expression

$$e_1(w/h,\epsilon,M) \simeq 5/3\ C_0(w/h,\epsilon,M,K) - 2\ C_0(w/h,\epsilon,M/2,K) + 1/3\ C_0(w/h,\epsilon,M/4,K). \tag{41}$$

Hence, if the preceding formula is substituted in (40), one

TABLE II

w / h		ε = 6.0	ε = 9.5	ε = 16.0	ε = 28.0
0.1	a	134.6181	109.4533	85.7565	65.5287
	b	135.455	110.172	85.9659	65.5298
	c	134.352	110.058	87.762	68.819
0.2	a	112.4178	91.3236	71.5062	54.6170
	b	113.272	91.809	71.6954	54.6138
	c	112.255	91.776	73.015	57.110
0.4	a	90.3169	73.2752	57.3199	43.7543
	b	91.172	73.702	57.4999	43.7391
	c	89.909	73.290	58.110	45.281
0.7	a	72.7336	58.9191	46.0375	35.1161
	b	73.613	59.379	46.2344	35.1153
	c	71.995	58.502	46.217	35.872
1.0	a	61.8381	50.0281	39.0528	29.7697
	b	62.713	50.501	39.2512	29.7629
	c	60.970	49.431	38.948	30.144
2.0	a	42.2609	34.0775	26.5368	20.1967
	b	43.149	34.592	26.7555	20.2086
	c	41.510	33.493	26.248	20.197
4.0	a	26.4336	21.2335	16.4884	12.5260
	b	27.301	21.763	16.7210	12.5529
	c	26.027	20.906	16.300	12.474
10.0	a	12.7125	10.1664	7.86887	5.96534
	b	13.341	10.568	8.0385	5.9746
	c	12.485	9.981	7.8079	5.892

Note: Characteristic impedances, obtained by this algorithm *a*, by the method of moments *b*, and by conformal mapping *c*.

obtains

$$C(w/h,\epsilon) \simeq \mathbf{C}(w/h,\epsilon,M,K) = 8/3\, C_0(w/h,\epsilon,M,K) - 2\, C_0(w/h,\epsilon,M/2,K) + 1/3\, C_0(w/h,\epsilon,M/4,K). \quad (42)$$

Then one can conclude that $\mathbf{C}(w/h,\epsilon,M,K)$ approximates the exact value $C(w/h,\epsilon)$ much more closely than $C_0(w/h,\epsilon,M,K)$ for the same value M; in this way the influence of error e_1 is drastically reduced.

Table I shows for $w/h = 1$ and $\epsilon = 6$, the value of $\mathbf{C}(w/h,\epsilon,M,K)$ normalized with respect to ϵ_0 as a function of M and K. One can easily observe the reduction of error when M and K increase. Furthermore it is convenient to choose the value of $K \simeq 2M$ in order that the errors due to the finiteness of M and K be of the same order. However, with $M = 4$ and $K = 8$, an accuracy 4×10^{-3} is obtained, while with $M = 64$ and $K = 128$ the accuracy is 10^{-7}. The same results are obtained with different values of w/h and ϵ. Finally the values in Table I confirm that the roundoff errors e_3 are negligible.

The computer time depends in a significant way only on M and N_s, according to the following formula (S in seconds):

$$S = a_1 N_s + a_2 N_s M + a_3 M^2 \quad (43)$$

where, on the IBM 360/67 computer the values of the coefficients, estimated experimentally, are

$$a_1 = 0.0007 \quad a_2 = 0.000048 \quad a_3 = 0.000098.$$

For example, for an accuracy of 10^{-3} the required time is about 10^{-2} s.

The characteristic impedance Z_0 of the microstrip is then calculated by using the expression $Z_0 = \mu_0/\{v_0 \cdot [C(w/h,1)C(w/h,\epsilon)]^{1/2}\}$ where μ_0 ($4\pi\ 10^{-7}$ H/m) is the magnetic permeability, and v_0 ($2.9978\ 10^8$ m/s) is the velocity of light. The computed results for a microstrip with different values of ϵ and w/h are given in Table II and compared with those obtained by Farrar and Adams [18] (by the method of moments) and by Sobol [22] (by conformal mapping). The values of the capacitance $C(w/h,\epsilon)$ were computed with $M = 32$ and $K = 64$ (accuracy of about 10^{-6}).

VII. CONCLUSIONS

A method was presented for the evaluation of TEM parameters of microstrips based on a philosophy entirely different from that followed in the current literature. The circuit model used allows one to obtain very high accuracies (10^{-8}) or to reduce the computer time when ordinary accuracies are allowed (10^{-2} s for a 10^{-3} accuracy). The model can be easily extended to cover more complicated structures.

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Calculation of Microstrip Discontinuity Inductances

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***Abstract*—Inductive components of microstrip discontinuity equivalent circuits are calculated by the Galerkin method. The formulation and method of calculation are discussed and a large number of numerical results for symmetric corners, T junctions, and steps changes are presented. These results compare well with experiment.**

INTRODUCTION

THE characterization of microstrip discontinuities by equivalent circuits is currently of some interest. Detailed knowledge of the parameters in these circuits enables easy implementation of paper designs without tedious cut-and-try methods. While the published literature [1]–[4] provides curves for the capacitive components of these circuits, little is available for their inductive components. The method of calculation suggested by Horton [5], [6] is not rigorous since the inductance calculation is based on charge estimates and the results obtained are not in agreement with experiment. The magnetic wall model has been used for triplate lines which are wide and homogenous and have confined fields, but its extension to microstrip lines which are inhomogenous and much narrower in comparison and are open structures is not completely justified.

Quasi-static calculation of inductance by the moment method [7] has provided results which show reasonable agreement with experiment. However, the disadvantage of this method for these three-dimensional problems as shown by Farrar and Adams [1] is the very large computer store requirements even for modest discretization. The alternative is to use a finite-element method as in the skin-effect formulation [8]. The results of discontinuity inductance from this method obtained previously were inaccurate as they were arrived at by subtracting two nearly equal numbers. Also the method was limited to finite-length strips and thus could not represent the actual situation in which the strips extend so far from the discontinuity that they may be considered as semi-infinite. The present paper is an extension of this finite-element method which overcomes these difficulties, and the results obtained for right-angled bends, step-width changes, and symmetrical T junctions are presented in the form of curves. Comparison of the results with experiment [9] shows reasonable

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